

Decoherence of a qubit by non-Gaussian noise at an arbitrary working point

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The decoherence of a qubit due to a classical non-Gaussian noise with correlation time longer than the decoherence time is discussed for arbitrary working points of the qubit. A method is developed that allows an exact formula for the phase memory functional in the presence of independent random telegraph noise sources to be derived.

I. INTRODUCTION

Reducing the decoherence induced by interaction with the environment is one of the major challenges in the practical implementation of quantum computing. In particular, for solid state qubits this seems to be the most important issue. In this paper we discuss the effect of noise with a correlation time that is long compared to the decoherence time of the qubit. The work is directly motivated by experiments^{1,2} on Josephson charge qubits (JCQ), but due to their general nature they can also be relevant to many other systems. The JCQ is built around a small superconducting grain connected to a superconducting reservoir by a Josephson junction. By means of a capacitively coupled gate voltage one can control the number of Cooper pairs on the small grain. Because of the Coulomb blockade there will be a preferred number of Cooper pairs except at special degeneracy points where the energies of states with n and $n + 1$ coincide. The degeneracy is lifted by the Josephson coupling, giving the usual level anticrossing picture sketched in Fig. 1. If the gate voltage is never too far from the degeneracy point we can ignore transitions to other levels and the device can be regarded as a two level system or qubit.

Consider noise in this system originating from fluctuations in the electric potential of the grain. This could be either due to fluctuations in the gate voltage source, or to fluctuating charges in the environment. Both would correspond to fluctuations of the working point position along the horizontal axis of the figure, with a corresponding change in the energy. The experiments¹ where conducted with the gate voltage away from the degeneracy point, where to a good approximation the energy is a linear function of the potential. This case is covered by the theory^{3,4} for the case of free induction and narrow distribution of coupling constants and for both free induction and echo at arbitrary distribution of coupling constants, respectively.

Using a modified circuit Devoret *et. al.*² were able to work at the degeneracy point where to first order there is no change in energy when the potential is changed. The idea is that this would make the device less sensitive to electrostatic noise. This is then called the *optimal working point*. At this point the expectation value of

the charge is the same for both states. The purpose of this paper is to extend the theory^{3,4} to this situation and to see how dephasing is changed as the optimal point is approached.

The paper is organized as follows. In section II the model for the qubit interacting with the noise is presented. In section III we discuss a short time expansion that illustrates in a simple way the interplay of the effect of several uncorrelated fluctuators at the optimal point. The main purpose of this section is providing a qualitative understanding, as the results are contained within the full solution presented in section IV.

II. MODEL

Consider the Hamiltonian of the Josephson qubit, c. f. with Ref. 5,

$$H = \frac{1}{2}\Delta\sigma_z - \frac{1}{2}E_J\sigma_x. \quad (1)$$

Here $\Delta = E_cC_g(V_g - V_g^{\text{opt}})/e$, where E_c is the charging energy, $(V_g - V_g^{\text{opt}})$ is the deviation in gate voltage from the optimal working point, C_g is the gate capacitance and e the electron charge. The energy levels as function of the gate potential are shown in Fig. 1. In principle, both

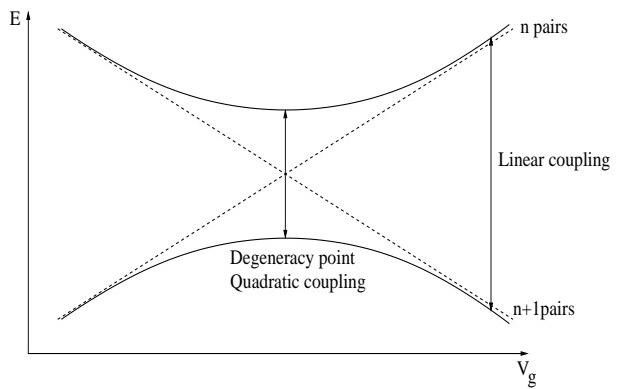


FIG. 1: Energy levels of Josephson qubit as function of the gate potential.

Δ and E_J can fluctuate in time, but for clarity we will

only consider the noise in Δ . Noise in E_J was considered in Ref. 6 and can be treated in a similar way.

We will describe fluctuations in Δ by a random time-dependent quantity $v(t)$. The Hamiltonian of the qubit is then

$$H = \frac{1}{2}[\Delta + v(t)]\sigma_z - \frac{1}{2}E_J\sigma_x. \quad (2)$$

One can express $v(t)$ through fluctuations of the effective gate voltage, $\delta V_g(t)$ as $v(t) = E_c C_g \delta V_g(t)/e$. One of the sources of such fluctuations is hopping background charges.^{3,4,7} We briefly recall the model of Refs. 3,4. It is assumed that there are dynamic defects characterized by two metastable states. They can be, for example, traps near the electrodes able to capture electrons from the electrodes and then re-emit them, or pairs of traps with one electron bouncing between them. Each dynamic defect is represented as a classical fluctuator producing random telegraph noise. That is, the state of each fluctuator is represented by a random function $\chi(t)$ which switches between the two values, $\pm 1/2$. The probability to switch n times between these states during the time t is assumed to be given by the Poisson distribution

$$P_n(t) = \frac{(\gamma t)^n}{n!} e^{-\gamma t}. \quad (3)$$

Here γ is the characteristic switching frequency of the given fluctuator. This means that the switching events occur independently of each other. The fluctuation $v(t)$ is a sum of the contributions of different fluctuators,

$$v(t) = \sum_i v_i \chi_i(t). \quad (4)$$

Different fluctuators are assumed to be statistically independent,

$$\langle \chi_i(t_1) \chi_k(t_2) \rangle = \frac{1}{4} e^{-2\gamma_i |t_1 - t_2|} \delta_{ik}. \quad (5)$$

This is actually the simplest model since it is assumed that the probabilities to jump between the states in two directions are the same. As a result, equilibrium populations of the states are equal. This assumption can be relaxed⁴ without change of the physical conclusions. Indeed, since only fluctuators with energy splittings $\lesssim kT$ contribute to the decoherence, the equilibrium populations of the efficient states do not essentially differ.

Several papers^{3,4,7} have addressed the dephasing in the presence of a large number of fluctuators with a broad range of switching rates, γ . This situation is relevant for systems showing $1/f$ noise. We believe, however, that the experiments with “good” qubits, e. g., Ref. 1, are best understood in terms of a small number of fluctuators. Also, if one is to achieve quantum computation this is necessary as a large number of fluctuators most probably will lead to large decoherence and ruin the device. We will therefore here focus on the effect of a small number of fluctuators.

The noise affects the qubit in two ways: (i) it causes shifts in the energy levels of the two states and thereby introduces a random contribution to the relative phase of the two states (dephasing) and (ii) it causes transitions between the two states leading to energy relaxation. Let us for the moment concentrate on the first effect, dephasing. The important quantity that we study is the phase-memory functional

$$\Psi = \left\langle \exp \left[\frac{i}{\hbar} \int_0^t dt' \beta(t') \delta E(\{\chi_i(t')\}) \right] \right\rangle \quad (6)$$

where $\delta E(\{\chi_i(t')\})$ is the shift in the energy splitting of the two levels caused by the fluctuators, while $\beta(t)$ is some function which depends on the qubit manipulation procedure, as will be explained below. The phase-memory functional describes the relative phase picked up during time evolution by one state of the qubit relative to the other.

Diagonalizing Eq. (2) we get the eigenenergies

$$E_{\pm} = \pm \frac{1}{2} \sqrt{(\Delta + v)^2 + E_J^2}. \quad (7)$$

If the working point is far from the optimal one, $\Delta \gg E_J$, we can neglect the Josephson energy, and coupling of the fluctuators to the qubit is linear,

$$E_{\pm} \approx \pm \frac{1}{2} \sqrt{\Delta^2 + E_J^2} \left(1 + \frac{v\Delta}{\Delta^2 + E_J^2} \right). \quad (8)$$

An exact formula for the phase memory functional for a linear coupling in the case of a single fluctuator coupled to the qubit was derived in Refs. 3,4,8. In this limit, because of the linearity E in v , the phase memory functional for any number of fluctuators is found by simply multiplying the phase memory functionals of different fluctuators. In the regime of exponential decay this procedure corresponds to simply adding the decay rates and the resulting decay rate is represented by the average over the distributions of fluctuator parameters.⁴ The optimal point, $\Delta = 0$, was studied in various approximations and numerically in Refs. 7,9.

The aim of the present work is to derive an exact formula for the phase memory functional of a single fluctuator similar to that in Refs. 3,4 that is applicable at an *arbitrary* working point. Averaging over a large number of fluctuators is in the general case not as simple as in the linear regime since the phase memory functional is not the product of individual factors for each fluctuator. However, we will extend the analysis to a small number of fluctuators, which we believe is relevant to qubit experiments.¹

To make the main idea clear let us first study the expansion at short times, $\gamma t \ll 1$. Firstly, this expansion provides some insight; secondly, the short-time situation can be most important for realistic qubits since at long times the phase memory functional has already decayed to a very low value.

III. PHASE MEMORY FUNCTIONAL AT SMALL TIMES, $\gamma t \ll 1$

We expand the energy in the limit

$$v \ll E_0 \equiv \sqrt{\Delta^2 + E_J^2}$$

to obtain

$$E_{\pm} = \pm \frac{1}{2} \left(E_0 + \frac{\Delta v}{E_0} + \frac{v^2}{2E^*} \right) \quad (9)$$

where $E^* \equiv E_0^3/E_J^2$. When averaging the phases we subtract the initial values so that we only get what comes from fluctuator jumps. Thus $\Psi \equiv \langle e^{i\phi(t)} \rangle$ where

$$\begin{aligned} \phi(t) = & \frac{1}{\hbar} \int_0^t \beta(t') dt' \left[\sum_i \frac{\Delta}{2E_0} v_i [\chi_i(t') - \chi_i^0] \right. \\ & \left. + \sum_{ij} \frac{v_i v_j}{4E^*} [\chi_i(t') \chi_j(t') - \chi_i^0 \chi_j^0] \right] \end{aligned} \quad (10)$$

is the random phase shift. Here $\chi_i^0 \equiv \chi_i(0)$ while the average is calculated over random telegraph processes in the fluctuators.

If there is only linear coupling, each fluctuator appears in only one term in the exponent. Since we assume the fluctuators to be statistically independent the average can be split into a product of averages over each individual fluctuator. With the quadratic term included this is no longer possible. Here we will study the cases of one and two fluctuators. This gives some insight to the general structure, and it is also the most relevant for realizing a working qubit since a large number of fluctuators would destroy the working of the qubit.

a. *Free induction signal:* Let us assume that $\beta(t) = \theta(t)$ where $\theta(t)$ is the Heaviside unit step function. This assumption corresponds to the free induction signal. With only one fluctuator the quadratic term is identically zero, and we have

$$\Psi_1 = \left\langle e^{i\nu \int_0^t dt' [\chi(t') - \chi^0]} \right\rangle, \quad \nu \equiv \frac{\Delta}{2E_0 \hbar} v. \quad (11)$$

Let us evaluate the memory functional in the limit of $\gamma t \ll 1$, first for one, and then for two fluctuators. Then the possibility of more than one jump is negligible, and we write for the probabilities of zero and one jump $P_0 = 1 - \gamma t$ and $P_1 = \gamma t$. The memory functional can be calculated by averaging over the time, t_1 , between the jumps, which at $\gamma t \ll 1$ are equally probable:

$$\Psi_1 = P_0 + \frac{P_1}{t} \int_0^t dt_1 \cos \nu(t - t_1) = 1 - \frac{\gamma}{\nu} (\nu t - \sin \nu t).$$

If $\nu t = (\Delta/2E_J \hbar)v t \ll 1$ we can expand the sine and get $\Psi_1 = 1 - \gamma \nu^2 t^3 / 6$. At the optimal point $\nu = 0$ and the fluctuator is not visible to the qubit. At other working points the dephasing rate is proportional to Δ^2 .

Now take the case of two fluctuators. We have to calculate $\Psi_2 \equiv \langle e^{i\phi_2(t)} \rangle$ where

$$\begin{aligned} \phi_2(t) = & \int_0^t dt' [\nu_1(\chi_1 - \chi_1^0) + \nu_2(\chi_2 - \chi_2^0) \\ & + 2\lambda_{12}(\chi_1 \chi_2 - \chi_1^0 \chi_2^0)] \end{aligned} \quad (12)$$

and $\lambda_{12} = v_1 v_2 / E^* \hbar$. Again, for $\gamma_i t \ll 1$ we get

$$\Psi_2 = 1 - \sum_{\pm, i=1,2} \left[\gamma_i t - \frac{\gamma_i}{2} \left(\frac{\sin(\nu_i \pm \lambda_{12}) t}{\nu_i \pm \lambda_{12}} \right) \right]. \quad (13)$$

Assuming $\nu_1, \nu_2, \lambda_{12} \ll t^{-1}$ and expanding sines we get:

$$\begin{aligned} \Psi_2 = & 1 - \sum_i \frac{\gamma_i}{6} (\nu_i^2 + \lambda_{12}^2) t^3 \\ \approx & \left(1 - \frac{\gamma_1 + \gamma_2}{6} \lambda_{12}^2 t^3 \right) \prod_i \left(1 - \frac{\gamma_i \nu_i^2 t^3}{6} \right). \end{aligned} \quad (14)$$

We see that in this limit one can split the expression for Ψ in factors corresponding to the individual fluctuators just as in the case of linear coupling, but there appears an additional factor due to the nonlinear coupling. It is easy to see that a similar pattern will also appear for larger number of fluctuators. At longer times this product structure is lost.

The interplay between the linear and quadratic coupling is now quite clear. If $\nu \gg \lambda$ the linear coupling is dominant. Approaching the optimal point will reduce the dephasing until ν becomes smaller than λ where the term proportional to λ^2 becomes most important. This contribution results from the interplay between the two fluctuators and cannot be eliminated; thus it represents the minimal dephasing possible at the optimal point. The physical reason for this is quite easy to understand. Switching of fluctuator 1 shifts the average point that fluctuator 2 is working around. Both positions of fluctuator 1 can not represent the optimal point with respect to fluctuator 2, and some dephasing is bound to occur. This is the most important physical insight that distinguishes the quadratic coupling from the linear. In the case of quadratic coupling, even if the different fluctuators in themselves are independent, their effect on the qubit will be influenced by the positions of all the others. With linear coupling one finds that slow fluctuators, with $\gamma t \ll 1$ do not contribute to the dephasing. This is no longer true for quadratic coupling, as they play a role in determining the effect of the fast fluctuators even if they do not have time to switch during the experiment. Thus very slow fluctuators may be of great importance.

b. *Two-pulse echo:* Perhaps more directly related to experiments are the echo signals. These are found using Eq. (10) where for two-pulse echo

$$\beta(t) = \begin{cases} +1 & \text{for } t < \tau, \\ -1 & \text{for } \tau < t < 2\tau. \end{cases} \quad (15)$$

Here τ is the delay between the initial pulse and the echo pulse, and the echo signal is centered around 2τ . For

short times ($\gamma\tau < 1$) this gives for one fluctuator

$$\Psi_1 = 1 - \frac{2\gamma}{\nu}(\nu\tau - \sin\nu\tau), \quad (16)$$

and for two fluctuators

$$\Psi_2 = 1 - \sum_{\pm, i=1,2} \left[2\gamma_i\tau - \gamma_i \left(\frac{\sin(\nu_i \pm \lambda_{12})\tau}{\nu_i \pm \lambda_{12}} \right) \right]. \quad (17)$$

IV. EXACT SOLUTION FOR SMALL NUMBER OF FLUCTUATORS

We now turn to the calculation of the phase memory functional for arbitrary times. The method described here is in principle applicable to an arbitrary number of fluctuators, but the resulting formulas quickly get impractically large when the number of fluctuators increase. The main idea is to consider the phase ϕ as a random variable with some probability distribution $p(\phi, t)$ that will depend on time. Once this is known the phase memory-function is given by $\Psi = \int dx e^{i\phi} p(\phi, t)$. By mapping to a correlated random walk problem we derive a Master equation for the probabilities $p(\phi, t)$. The details of how to calculate $p(\phi, t)$ are given in appendix A.

A. Distribution function for one fluctuator far from the optimal point

Free-induction signal: Let us first discuss the results for the distribution function in the simplest case, that of one fluctuator far from the optimal point. The phase memory functional for this problem was derived in Refs. 3,4, and in the end we will rederive the same

expression. However, it is the simplest example for illustrating the general method, and it gives new insight to the relation between the Gaussian approximation and the fluctuator model. To better understand the meaning of the results it is useful to recall the standard picture of dephasing by a *Gaussian* noise that is well known from NMR-physics (see, e. g., Ref. 10). If the time t entering the phase memory functional

$$\Psi = \langle e^{i\phi(t)} \rangle, \quad \phi(t) = \int_0^t dt' \nu(t'), \quad \nu(t) \equiv \nu \chi(t)$$

is much longer than the correlation time $\approx \gamma^{-1}$ of the fluctuating function $\nu(t)$, the integral can be considered as the sum of a large number of uncorrelated contributions. Consequently, by the central limit theorem, the phase will be distributed according to a Gaussian

$$p(\phi) = \frac{1}{\sqrt{2\pi\langle\phi^2\rangle}} e^{-\frac{\phi^2}{2\langle\phi^2\rangle}}$$

and the phase memory function is

$$\Psi = e^{-\langle\phi^2\rangle/2}. \quad (18)$$

From Eq. (5) for the correlation function we get

$$\langle\phi^2\rangle = \frac{\nu^2}{4\gamma}t + \frac{\nu^2}{2\gamma^2}(e^{-2\gamma t} - 1) \approx \frac{\nu^2}{4\gamma}t \text{ at } \gamma t \gg 1.$$

Thus, in the Gaussian approximation, the phase memory functional decays exponentially at $\gamma t \gg 1$ with the rate

$$\Gamma_\phi^{(G)} = \nu^2/8\gamma. \quad (19)$$

Using the method explained in Appendix A we can find an exact solution for the distribution function of ϕ

$$p(\phi, t) = e^{-\gamma t} \left[\delta(\phi \mp \nu t/2) + \frac{\gamma}{\nu} \frac{(t \pm 2\phi/\nu) I_1 \left(\gamma \sqrt{t^2 - (2\phi/\nu)^2} \right)}{\sqrt{t^2 - (2\phi/\nu)^2}} \right] \left[\theta \left(\frac{2\phi}{\nu} + t \right) - \theta \left(\frac{2\phi}{\nu} - t \right) \right] \quad (20)$$

where the different signs correspond to different initial states of the fluctuator and $I_1(z)$ is the modified Bessel function. Without jumps of the fluctuator, the result would be only the moving δ -pulse of a constant amplitude, $\delta(\phi \mp \nu t/2)$. The value $\nu t/2$ is the maximal possible value of ϕ acquired for the time t , while the jumps of the fluctuators account for the smooth part. Averaging over the initial state of the fluctuator we get (not writing

the θ -functions)

$$p(\phi, t) = e^{-\gamma t} \left[\frac{\delta(\phi - \nu t/2) + \delta(\phi + \nu t/2)}{2} + \frac{\gamma}{\nu} \cdot \frac{I_1(\gamma t \sqrt{1 - (2/\nu t)^2 \phi^2})}{\sqrt{1 - (2/\nu t)^2 \phi^2}} \right]. \quad (21)$$

This is plotted in Fig. 2 for the times $t = 1, 5, 10$, $\gamma = 1$ and $\nu = 1$. We observe that the central region is similar to a Gaussian, but at short times this is cut off by the δ -functions represented by the vertical lines. At $\gamma t \gg 1$ the function is indeed close to a Gaussian, as can be seen

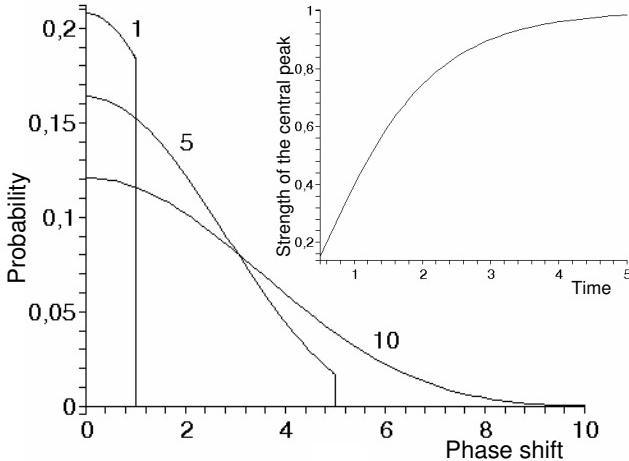


FIG. 2: Distribution function at time 1, 5, 10 for $\gamma = 1$ and $\nu = 1$. Inset: time dependence of the integrated strength of the central peak.

from the asymptotic expansion of the Bessel function

$$e^{-\gamma t} \frac{I_1(\gamma t \sqrt{1 - (2/\nu t)^2 \phi^2})}{2\sqrt{1 - (2/\nu t)^2 \phi^2}} \sim \frac{1}{\sqrt{2\pi\gamma t}} e^{-\frac{2\gamma\phi^2}{\nu^2 t}}$$

Comparing to Eq. (18) we see that we recover the result for the dephasing rate in Eq. (19).

However, we know from Ref. 4 that if $\nu > 2\gamma$ we have pronounced non-Gaussian behavior. We can now understand this from the point of view of the distribution function. The smooth central part of this indeed approaches a Gaussian for $\gamma t \gg 1$ and this gives the dephasing rate (19), but the δ -functions at the ends only decay at the rate γ . As long as the $\Gamma_\phi^{(G)}$ of (19) is smaller than γ the decay will be controlled by the central part and the Gaussian approximation is valid. If $\Gamma_\phi^{(G)} > \gamma$ the decay is limited by the δ -functions, and is set by the rate γ .

Using the distribution function (21) one can calculate the phase memory functional (see Appendix A) and one finds for $\gamma t \ll 1$ an exponential decay with rate

$$\Gamma_\phi = \gamma - \text{Re} \left(\sqrt{\gamma^2 - \nu^2/4} \right). \quad (22)$$

Figure 3 shows Γ_ϕ as function of ν at $\gamma = 1$ for the Gaussian and the fluctuator models.

Two-pulse echo: The same method as outlined in the Appendix A allows one to calculate the phase distribution function, $p_e(\phi, \tau)$ for the two-pulse echo signal. The result can be expressed in the form

$$p_e(\phi, \tau) = e^{-2\gamma\tau} \left[\delta(\phi) + \frac{\gamma}{\pi} \int_0^\infty dk \frac{\cos k\phi \sin w_k \tau (\gamma \sin w_k \tau + w_k \cos w_k \tau)}{w_k^2} \right].$$

Here $w_k \equiv \sqrt{(k\nu/2)^2 - \gamma^2}$, while τ is the delay time between the first and second pulse. The smooth part of

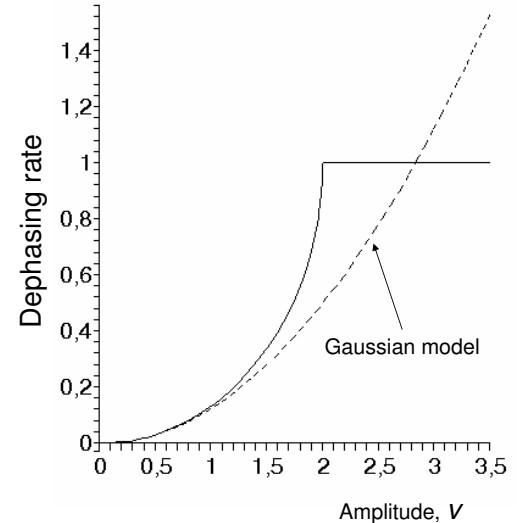


FIG. 3: Dephasing rate as function of ν , $\gamma = 1$. The solid line is the exact result for one fluctuator, Eq. (22), and the dashed line is the Gaussian approximation, Eq. (19).

the distribution given by the second item in the above formula is plotted in Fig. 4. It is qualitatively similar to

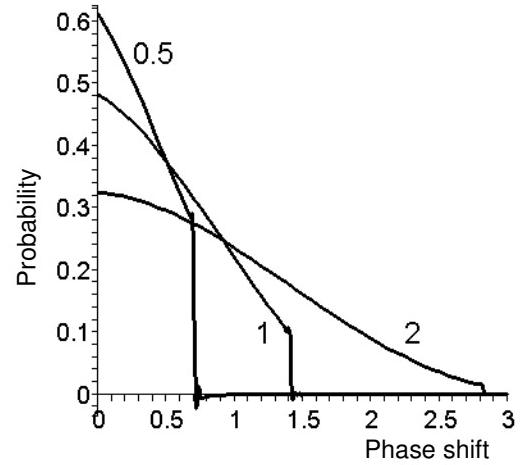


FIG. 4: Distribution of the phases for the echo signal for $\nu = \gamma$ and different products $\gamma\tau$ (shown near the curves).

the phase distribution for the free induction.

B. Dephasing rate as function of working point

Let us then consider the decay of the phase memory functional *for different working points*. Again we recall the situation in NMR-physics where the loss of the signal after the spins are set precessing by a $\pi/2$ -pulse is caused by two independent processes. The phase memory functional considered above measures the random contributions to the phase caused by fluctuating energy difference

between the two states (which in an NMR experiment is caused by fluctuations in the magnetic field parallel to the external main field). In addition there are processes which flip the spin from one state to another, so called T_1 -processes. These are caused by fluctuations in the magnetic field normal to the external field. If we denote the decay rate of the excited state into the ground state Γ_1 and add the two contributions we have the total decay of the spin precession signal

$$\Gamma_2 = \frac{1}{2}\Gamma_1 + \Gamma_\phi. \quad (23)$$

The factor $1/2$ in front of Γ_1 can be understood from the fact that if the probability of the excited state decays with rate Γ_1 , the amplitude decays with the rate $\Gamma_1/2$ and this is what enters the off-diagonal elements of the density matrix. For a more detailed discussion see Ref. 10, or, for qubits, Ref. 9.

Let us now discuss how the relative strength of the two terms of Eq. (23) changes as we change the working point of the qubit. Looking back at the Hamiltonian (2) we remember that it is equivalent to a spin $1/2$ -particle in a static magnetic field $\mathbf{B} = E_J\mathbf{e}_x + \Delta\mathbf{e}_z$ while the noise is always along the z -axis, $\nu = \nu\mathbf{e}_z$. We denote the angle between \mathbf{B} and the z -axis by $\theta = \arctan(E_J/\Delta)$. In particular, $\theta = 0$ corresponds to working far from the degeneracy point where $\delta \gg E_J$ while $\theta = \pi/2$ is the degeneracy (optimal) point $\Delta = 0$. The time evolution of the qubit is then a precession on the Bloch sphere around the total field (Fig. 5). Far from the degeneracy

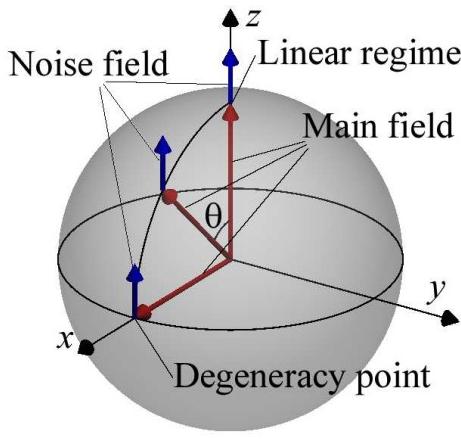


FIG. 5: Different working points represented on the Bloch sphere

point the main field is directed along the z -axis (pointing at the north pole of the sphere), whereas at degeneracy it is along the x -axis (on the equator). All the time the noise vector v is in the z -direction. We see that the noise component parallel to the external field, which gives the Γ_ϕ , is largest far from degeneracy, which agrees with our previous discussion. However, the noise normal to the field, giving T_1 -processes, is maximal at degeneracy. In

the Gaussian approximation this is given by (see Ref. 10)

$$\Gamma_1 \equiv \frac{1}{T_1} = \sin^2 \theta \int_0^\infty d\tau \langle \nu(t)\nu(t+\tau) \rangle \cos \frac{E_0\tau}{\hbar}.$$

Using the correlator (5) one can rewrite this expression for the case of N identical fluctuators with parameters ν and γ as

$$\Gamma_1 = \frac{N}{2} \frac{\hbar^2 \nu^2 \gamma}{E_0^2 + 4\gamma^2} \left(\frac{E_J}{E_0} \right)^2. \quad (24)$$

The Gaussian approximation for the Γ_ϕ is more difficult to obtain because the square root in the energy (7) makes the average in Eq. (18) not treatable analytically. However we can expand in the lowest order in v/E_0 , which is a good approximation except a close vicinity of the degeneracy point, where the coefficient in front goes to zero and higher order terms need to be calculated. This gives the same result as the $\Gamma_2^{\text{ad}} = 1/T_2^{\text{ad}}$ of Ref. 10,

$$\Gamma_\phi^{(G)} \approx \frac{1}{2} \cos^2 \theta \int_0^\infty d\tau \langle \nu(t)\nu(t+\tau) \rangle,$$

which for N identical uncorrelated fluctuators yields

$$\Gamma_\phi^{(G)} = \frac{N}{8} \frac{\nu^2}{\gamma} \left(\frac{\Delta}{E_0} \right)^2. \quad (25)$$

Now we want to compare these Gaussian results with the exact expressions found using the method of Appendix A. In Figure 6 the relaxation times (inverses of the relax-

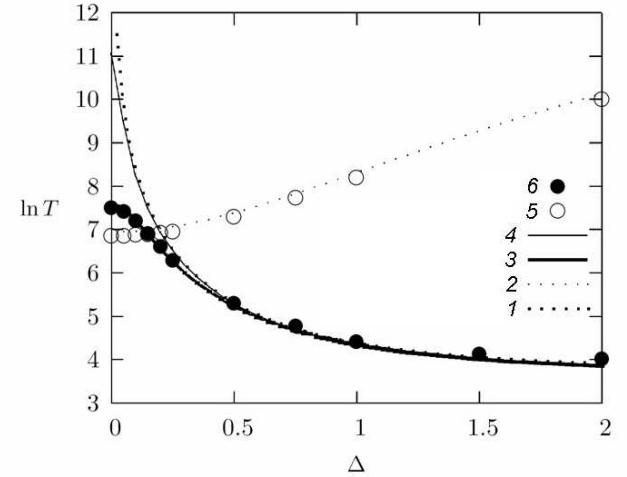


FIG. 6: Relaxation times for the case of two fluctuators as function of working point for weak coupling, $v/E_J = 0.1$, and $\gamma/E_J = 0.1$. Δ is measured in units of E_J . 1 – $1/\Gamma_\phi^{(G)}$, Eq. (25), 2 – T_1 , Eq. (24), 3 – $T_2 \equiv (1/2T_1 + 1/T_\phi)^{-1}$, 4 – T_ϕ , calculated according to exact expressions from Appendix A. 5, 6 – results of numerical simulation for T_1 and T_2 , respectively

ation rates) T_1 (curve 2) and $1/\Gamma_\phi^{(G)}$ (curve 1) are shown.

They are calculated according to Eqs. (24) and (25), respectively. The decay time, $T_\phi = \Gamma_\phi^{-1}$ (curve 4), of the phase memory functional calculated using the method of Appendix A and the resulting decay time, T_2 (curve 3), of the spin signal according to Eq. (23) are also shown. The points represent the rates obtained by a numerical simulation of the time evolution according to the Hamiltonian (2), which performs averaging over many realizations of the random process. All curves are calculated for the case of 2 fluctuators with coupling strength $\nu/E_J = 0.1$ and switching rate $\gamma/E_J = 0.1$. Thus $v/\gamma = 1$ and this case belongs to the so-called weak coupling regime.⁴ We see that for all working points the decay is well described by the Gaussian approximation. This is because the rates always are slower than the limiting rate γ set by the correlation time of the fluctuators, similar to what was described above for the case far from the degeneracy point. For most working points the rate $\Gamma_\phi \gg \Gamma_1$ and this dominates the Γ_2 . Close to the degeneracy point we see that Γ_1 becomes more important and at degeneracy it dominates completely giving $\Gamma_2 \approx \Gamma_1/2$. The same quantities

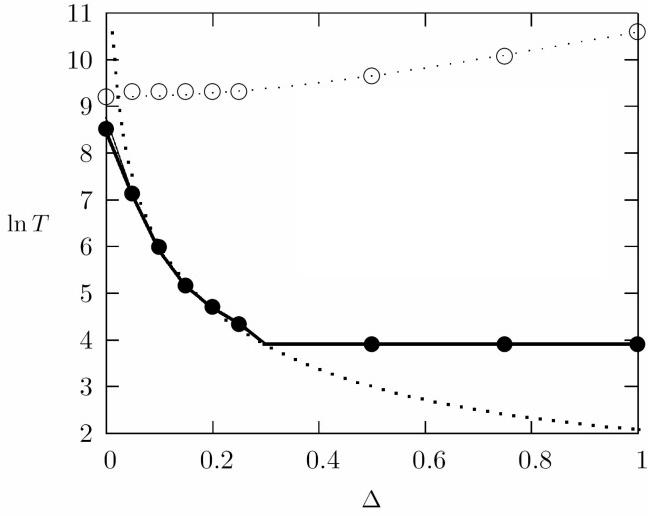


FIG. 7: Relaxation times as function of working point for the case of strong coupling. Two fluctuators with $v/E_J = 0.1$, $\gamma/E_J = 0.01$. The legend is the same as in Fig. 6

for the case of strong coupling $v/E_J = 0.1$, $\gamma/E_J = 0.01$, $v/\gamma = 10$ are shown in Figure 7. Far from the degeneracy the situation is similar to the one described earlier, with the rate, γ , determined by the decay of the δ -function peaks in the distribution function. Closer to the degeneracy the rate is slower and the Gaussian approximation gives good results.

Note that in the case of strong coupling the difference between the approximate expression $\Gamma_\phi^{(G)}$ and the exact Γ_ϕ becomes more noticeable. At large Δ this is because of the essentially non-Gaussian character of the noise, as discussed before. At small Δ the difference comes from the fact that the $\Gamma_\phi^{(G)}$ only gives the Gaussian approximation to Γ_ϕ in the lowest order in ν/E_J . We expect that

if we had calculated the phase memory function in the Gaussian approximation according to (18) and using the exact expectation value $\langle \phi^2 \rangle$ (which is hard to find analytically) the result would agree completely with the Γ_ϕ calculated by the method of Appendix A since the decay time is much longer than the correlation time $1/\gamma$ of the noise and the central limit theorem should work. Note also that at degeneracy the decay is still dominated by the phase relaxation processes, Γ_ϕ , while the T_1 -processes only give a small correction.

Figures 6 and 7 should be compared to the experimental results of Astafiev *et al.*¹¹ There is clear qualitative agreement, but it is hard to try to make a quantitative fit, especially for the T_2 where there is very little data.

Let us rather look back at the Hamiltonian (2) and Figure 5. So far we only considered noise in the σ_z component of the Hamiltonian, as appropriate for noise sources coupled to the charge of the qubit. As discussed in Ref. 6 there is also the possibility of noise in the Josephson coupling (the σ_x part). In Figure 5 this would correspond to a noise vector parallel to the x -axis. This noise would be transversal far from degeneracy, giving large T_2 and small T_1 and it would be longitudinal at degeneracy point, with T_1 large and T_2 small. The fact that the experimental results are similar to our figures 6 and 7 rather than the opposite shows that in these experiments the noise in the σ_z part is dominant.

V. DISCUSSION

At the end we would like to discuss two simple observations based on the above results.

A. Measuring the distribution function?

The method that we have used to find the phase memory functional gave as an intermediate result an expression for the distribution function of the phase $p(\phi, t)$. The phase memory functional is the average of the quantity $e^{i\phi}$ with this distribution. But can the full distribution function be compared to experiment? In the first set of experiments¹ only averages could be measured, but recently there has been demonstrated single shot readout of the qubit state.¹² Is it possible to extract the full distribution function from such data? Unfortunately, the answer is no for the following reason. For each realization of the experiment, there is a certain realization of the random external noise. This gives the phase difference ϕ a certain well defined value. The next realization gives another realization of the noise process and another value for the phase difference and so on. Since we have no knowledge about which realization of the noise is relevant to a certain experiment, we have to describe the final state of the qubit by a density matrix representing a mixed state. Since all outcomes of all possible experiments can be calculated from the density matrix, no

more information about the system can be learned that what is contained in the density matrix. Since the phase distribution was calculated ignoring the relaxation (T_1) processes the density matrix will be represented by a vector in the equatorial plane of the Bloch sphere (but for a mixed state it will not be on the surface of the sphere but at some interior point), and the components of this vector are exactly the real and imaginary parts of the phase memory functional. Thus, the phase memory functional contains all information that we can extract about the qubit through experiments. A different way to express the same is that there are many different phase distribution functions yielding the *same* qubit density matrix, and there is no way that one can experimentally distinguish these in an experiment with a single qubit.

B. Non-Gaussian T_1 ?

Notice that it seems from the figures 6 and 7 that the energy relaxation time T_1 is always well described by the Gaussian approximation. Can we understand this in a better way? The Gaussian approximation is good provided the correlation time of the noise is much shorter than the decay time, T_1 or T_2 . In that case separation of timescales enables one to deduce an exponential decay with rates:¹⁰

$$1/T_1 \propto S(E_0), \quad 1/T_2 = 1/2T_1 + \Gamma_2^{ad}, \quad \Gamma_2^{ad} \propto S(0)$$

where $S(\omega)$ is the noise power spectrum. For the telegraph process with switching rate γ and coupling strength v we find that the time correlation is given by

$$\langle \chi(t)\chi(0) \rangle = e^{-2\gamma|t|}$$

so the correlation time is $1/2\gamma$ and the noise power spectrum

$$S(\omega) \propto \frac{\gamma v^2}{\gamma^2 + \omega^2}.$$

This gives the relative rates

$$\frac{1}{T_1\gamma} \propto \frac{v^2}{\gamma^2 + E_J^2} \leq \left(\frac{v}{E_J}\right)^2, \quad \frac{\Gamma_2^{ad}}{\gamma} \propto \left(\frac{v}{\gamma}\right)^2.$$

We see that as long as the noise is weak compared to the qubit splitting, $v < E_0$, the condition $\gamma T_1 \gg 1$ for the validity of the Gaussian approximation, is always satisfied. For the T_2 the situation is different, and the condition $\Gamma_2 < \gamma$ is violated if $v > \gamma$, leading to non-Gaussian behavior⁴. So in the case of dephasing the Gaussian approximation so to speak predicts its own breakdown whereas for energy relaxation it is consistent as long as the noise is weak compared to the level spacing.

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APPENDIX A: CALCULATION OF DISTRIBUTION FUNCTION

1. Single fluctuator

To explain the method we re-derive the result for one fluctuator with linear coupling since this is the simplest case. Let us for simplicity also assume that $\beta(t) = 1$ at $t > 0$ and start with the calculation of $\Psi_1(t)$.

Let us discretize the integral (12) for $\lambda_{12} = 0$ introducing small time steps $\tau \equiv t/N$, where $N \gg 1$. Then the random phase shift $\phi(t)$ can be expressed as

$$\phi(t) = \nu\tau \sum_{n=1}^N \chi_n, \quad \chi_n \equiv \chi(n\tau).$$

Hence, the integration over time can then be thought of as a random walk process, where at each time step the random walker moves a step $\sigma = \tau\nu/2$ in the direction depending on the current position of the fluctuator. The steps are correlated, but only with the previous step. The probability that a step is in the same direction as the previous one is $\alpha = 1 - \gamma\tau$ and the probability for a step to be in the opposite direction is $\beta = \gamma\tau$. Let m be the number of steps from the origin (so that the position is $x = \sigma m$). We want to find the probability $P_n(m)$ to be in position m at time step n (dimensional time $t_n = n\tau$). This is found by the following method. We split the probability in two parts: the probability to reach point m coming from the right, $A_n(m)$ and from the left $B_n(m)$, so that $P_n(m) = A_n(m) + B_n(m)$. We then have the equations

$$\begin{aligned} A_{n+1}(m) &= \alpha A_n(m-1) + \beta B_n(m-1), \\ B_{n+1}(m) &= \beta A_n(m+1) + \alpha B_n(m+1). \end{aligned} \quad (\text{A1})$$

We need the continuum limit, letting $N \rightarrow \infty$, and $\tau \rightarrow 0$ with $N\tau = t$ fixed. Writing

$$\begin{aligned} a(\phi, t) &= a(m\sigma, n\tau) = A_n(m), \\ b(\phi, t) &= b(m\sigma, n\tau) = B_n(m) \end{aligned} \quad (\text{A2})$$

and expanding to first order in τ we get

$$\begin{aligned} a + \tau a_t &= \alpha(a - \sigma a_\phi) + \beta(b - \sigma b_\phi), \\ b + \tau b_t &= \beta(a + \sigma a_\phi) + \alpha(b + \sigma b_\phi). \end{aligned} \quad (\text{A3})$$

Here subscripts ϕ and t denote partial derivatives with respect to ϕ and t , respectively. Adding and subtracting these we get (with $p = a + b$ and $q = a - b$)

$$\begin{aligned} \tau p_t &= (\beta - \alpha)\sigma q_\phi, \\ q + \tau q_t &= (\alpha - \beta)q - \sigma p_\phi. \end{aligned} \quad (\text{A4})$$

Differentiating the second of these and inserting q_ϕ from the first we obtain the final equation for p

$$p_{tt} + 2\gamma p_t = \left(\frac{\nu}{2}\right)^2 p_{\phi\phi} \quad (\text{A5})$$

which is called the telegraph equation. We guess the solution $p = e^{i(k\phi - \omega t)}$ and get the dispersion relation

$$\omega_{\pm} = -i\gamma \pm \sqrt{\left(\frac{\nu}{2}\right)^2 \kappa^2 - \gamma^2}. \quad (\text{A6})$$

The general solution is then

$$p(\phi, t) = \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi} [a_{\kappa} e^{-i\omega_+ t} + b_{\kappa} e^{-i\omega_- t}] e^{i\kappa\phi}. \quad (\text{A7})$$

The coefficients $\{a_{\kappa}, b_{\kappa}\}$ can be obtained from the initial conditions,

$$\begin{aligned} \delta(\phi) &= p(\phi, 0) = \int_{-\infty}^{\infty} d\kappa [a_{\kappa} + b_{\kappa}] e^{i\kappa\phi} \\ &\implies a_{\kappa} + b_{\kappa} = 1, \end{aligned} \quad (\text{A8})$$

$$\begin{aligned} \nu\chi_0\delta'(\phi) &= p_t(\phi, t) = -i \int_{-\infty}^{\infty} d\kappa [\omega_+ a_{\kappa} + \omega_- b_{\kappa}] e^{i\kappa\phi} \\ &\implies \omega_+ a_{\kappa} + \omega_- b_{\kappa} = -\kappa\nu\chi_0, \end{aligned} \quad (\text{A9})$$

which yield

$$\{a_{\kappa}, b_{\kappa}\} = \frac{1}{2} \mp \frac{\kappa\nu\chi_0 - i\gamma}{2\sqrt{\kappa^2\nu^2/4 - \gamma^2}}. \quad (\text{A10})$$

This is to be inserted into Eq. (A7). The result is given by Eq. (20).

The memory functional can be expressed as the expectation value $\Psi_1 = \int d\phi p(\phi, t) e^{i\phi}$. Thus

$$\begin{aligned} \Psi_1 &= \int_{-\infty}^{\infty} d\kappa [a_{\kappa} e^{-i\omega_+ t} + b_{\kappa} e^{-i\omega_- t}] \delta(\kappa + 1) \\ &= \frac{1}{2\mu} e^{-\gamma t} \sum_{\pm} \left(\mu \pm 1 \pm \frac{i\nu\chi_0}{\gamma} \right) e^{\pm\mu\gamma t} \end{aligned} \quad (\text{A11})$$

$$\begin{aligned} A_{n+1}(\phi) &= \alpha^2 A_n(\phi - \bar{\alpha}\tau) + \beta^2 B_n(\phi - \bar{\alpha}\tau) + \alpha\beta C_n(\phi - \bar{\alpha}\tau), \\ B_{n+1}(\phi) &= \beta^2 A_n(\phi - \bar{\beta}\tau) + \alpha^2 B_n(\phi - \bar{\beta}\tau) + \alpha\beta C_n[\phi - \bar{\beta}\tau], \\ C_{n+1}(\phi) &= 2\alpha\beta A_n(\phi - \bar{\gamma}\tau) + 2\alpha\beta B_n(\phi - \bar{\gamma}\tau) + (\alpha^2 + \beta^2) C_n(\phi - \bar{\gamma}\tau). \end{aligned} \quad (\text{A12})$$

Again we introduce continuous variables a, b, c and expand to first order in τ to get

$$\begin{aligned} a_t &= -2\gamma a - \bar{\alpha}a_{\phi} + \gamma c, \\ b_t &= -2\gamma b - \bar{\beta}b_{\phi} + \gamma c, \\ c_t &= 2\gamma a + 2\gamma b - 2\gamma c - \bar{\gamma}c_{\phi}. \end{aligned} \quad (\text{A13})$$

This can be written in matrix form

$$\mathbf{a}_t = M\mathbf{a}, \quad \mathbf{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (\text{A14})$$

where $\mu = \sqrt{1 - \nu^2/4\gamma^2}$. This agrees with the result of Ref. 4.

2. Two fluctuators

Now we turn to the case of two fluctuators. Again we discretize (12) to get

$$\phi_2 = \tau \sum_{n=1}^N (\nu\chi_1 + \nu\chi_2 + 2\lambda\chi_1\chi_2).$$

There are now three kinds of steps, depending on the settings of the fluctuators. If both have the value $+\frac{1}{2}$ there is a step $\bar{\alpha} = \nu + \frac{\lambda}{2}$, if both are $-\frac{1}{2}$ there is a step $\bar{\beta} = -\nu + \frac{\lambda}{2}$, and for one of each the step is $\bar{\gamma} = -\frac{\lambda}{2}$. We have the following probabilities for each jump, depending on the previous state:

	$\bar{\alpha}$	$\bar{\gamma}$	$\bar{\beta}$
$\bar{\alpha}$	α^2	$2\alpha\beta$	β^2
$\bar{\gamma}$	$\alpha\beta$	$\alpha^2 + \beta^2$	$\alpha\beta$
$\bar{\beta}$	β^2	$2\alpha\beta$	α^2

The total probability must now be split in 3 parts $P_n(\phi) = A_n(\phi) + B_n(\phi) + C_n(\phi)$, where $A_n(\phi)$ is the probability to reach point ϕ at time step n with a $\bar{\alpha}$ jump, $B_n(\phi)$ with a $\bar{\beta}$ jump and $C_n(\phi)$ with a $\bar{\gamma}$ jump. We have then the set of equations

where

$$M = \begin{pmatrix} -2\gamma - \bar{\alpha}\partial_{\phi} & 0 & \gamma \\ 0 & -2\gamma - \bar{\beta}\partial_{\phi} & \gamma \\ 2\gamma & 2\gamma & -2\gamma + \bar{\gamma}\partial_{\phi} \end{pmatrix}. \quad (\text{A15})$$

We then guess the solution in the form

$$\mathbf{a} = \mathbf{A}e^{i(\kappa\phi - \omega t)} \quad (\text{A16})$$

which gives the eigenvalue equation

$$-i\omega\mathbf{a} = \tilde{M}\mathbf{A} \quad (\text{A17})$$

where \tilde{M} is the matrix M with ∂_ϕ replaced by $i\kappa$. From this we get the dispersion equation for ω in terms of κ :

$$\begin{aligned} \omega^3 + \left(6\gamma i - \kappa \frac{\lambda}{2}\right) \omega^2 \\ + \left(-\kappa^2 \nu^2 - 8\gamma^2 - 4\gamma i \kappa \frac{\lambda}{2} - \kappa^2 \frac{\lambda^2}{4}\right) \omega \\ - 2\gamma i \kappa^2 \left(\nu^2 + \frac{\lambda^2}{4}\right) - \kappa^3 \frac{\lambda}{2} \left(\nu^2 - \frac{\lambda^2}{4}\right) = 0. \end{aligned}$$

This equation has the three solutions: ω_0 and ω_{\pm} where ω_0 is the solution that goes continuously to $-2i\gamma$ when $\lambda \rightarrow 0$. The general solution is then

$$\mathbf{a} = \int \frac{d\kappa}{2\pi} \tilde{A}_\kappa \begin{pmatrix} e^{-i\omega_0 t} \\ e^{-i\omega_+ t} \\ e^{-i\omega_- t} \end{pmatrix} e^{i\kappa\phi}, \quad (\text{A18})$$

where

$$\tilde{A}_\kappa = \begin{pmatrix} a_\kappa^a & b_\kappa^a & c_\kappa^a \\ a_\kappa^b & b_\kappa^b & c_\kappa^b \\ a_\kappa^c & b_\kappa^c & c_\kappa^c \end{pmatrix} \quad (\text{A19})$$

is a matrix of coefficients that has to be determined by the initial conditions.

Since we have three coefficients to determine for each of the a, b and c it appears that we need to specify both the

functions $a(\phi, 0)$... and the first two derivatives. However, because of the special form of the equations we can calculate all derivatives at $t = 0$ from the functions $a(\phi, 0)$...

$$\mathbf{a}_t(\phi, 0) = \tilde{M} \mathbf{a}(\phi, 0), \quad \mathbf{a}_{tt}(\phi, 0) = \tilde{M}^2 \mathbf{a}(\phi, 0). \quad (\text{A20})$$

The typical initial conditions would then correspond to specifying the initial type of jump in the random walk. For example if this was of type A we would have $a(x, 0) = \delta(x)$ and $b(x, 0) = c(x, 0) = 0$. Note that in this simple case where only one of the a, b and c are nonzero at $t = 0$ the procedure could be simplified by writing the general solution for $p = a + b + c$ and initial conditions for this. The more general case would be that all of a, b and c are nonzero and the complete matrix A_κ is needed. This would be the case for example when calculating echo signals, where the equations after the echo pulse has to be solved with initial conditions of this type corresponding to the solution of the equations before the echo pulse is applied.

Introducing the Fourier transformed functions (in ϕ)

$$\tilde{\mathbf{a}}_\kappa = \begin{pmatrix} \tilde{a}_\kappa \\ \tilde{b}_\kappa \\ \tilde{c}_\kappa \end{pmatrix} = \tilde{A}_\kappa \begin{pmatrix} e^{-i\omega_0 t} \\ e^{-i\omega_+ t} \\ e^{-i\omega_- t} \end{pmatrix} \quad (\text{A21})$$

we can write the initial conditions as

$$\tilde{\mathbf{a}}_\kappa(t = 0) = \tilde{A}_\kappa \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \tilde{M} \tilde{\mathbf{a}}_\kappa(t = 0) = -i A_\kappa \begin{pmatrix} \omega_0 \\ \omega_+ \\ \omega_- \end{pmatrix}, \quad \tilde{M}^2 \tilde{\mathbf{a}}_\kappa(t = 0) = -\tilde{A}_\kappa \begin{pmatrix} \omega_0^2 \\ \omega_+^2 \\ \omega_-^2 \end{pmatrix}. \quad (\text{A22})$$

These can be written more compactly if we introduce the matrix \tilde{M}_a with the left hand sides of the above equations as columns

$$\underbrace{\left(\tilde{\mathbf{a}}_\kappa(t = 0) \mid \tilde{M} \tilde{\mathbf{a}}_\kappa(t = 0) \mid \tilde{M}^2 \tilde{\mathbf{a}}_\kappa(t = 0) \right)}_{\tilde{M}_a} = A_\kappa \underbrace{\begin{pmatrix} 1 & -i\omega_0 & (-i\omega_0)^2 \\ 1 & -i\omega_+ & (-i\omega_+)^2 \\ 1 & -i\omega_- & (-i\omega_-)^2 \end{pmatrix}}_{\Omega} \quad (\text{A23})$$

From which the coefficients are found as $A_\kappa = \tilde{M}_a \Omega^{-1}$.

The final solution is then

$$p(\phi, t) = \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi} [a_\kappa e^{-i\omega_0 t} + b_\kappa e^{-i\omega_+ t} + c_\kappa e^{-i\omega_- t}] e^{i\kappa\phi}$$

where $a_\kappa = \sum_i a_\kappa^i$ and similarly for b_κ and c_κ . Again the average is calculated from

$$\begin{aligned} \langle e^{i\phi} \rangle &= \int d\phi e^{i\phi} p(\phi, t) \\ &= a_{-1} e^{-i\omega_0 t} + b_{-1} e^{-i\omega_+ t} + c_{-1} e^{-i\omega_- t} \end{aligned}$$

where $\kappa = -1$ in the ω in the last expression because of the δ -function from the ϕ integral.

Let us find the explicit expressions for the coefficients a_κ , b_κ and c_κ in this case. Adding the lines in Eq. (A23)

we get

$$\begin{aligned} a_\kappa + b_\kappa + c_\kappa &= 1, \\ a_\kappa \omega_0 + b_\kappa \omega_+ + c_\kappa \omega_- &= \kappa(\nu(\chi_1^0 + \chi_2^0) + 2\lambda\chi_1^0\chi_2^0) \equiv A_\kappa \\ a_\kappa \omega_0^2 + b_\kappa \omega_+^2 + c_\kappa \omega_-^2 &= \kappa^2(\nu(\chi_1^0 + \chi_2^0) + 2\lambda\chi_1^0\chi_2^0)^2 \\ -2i\gamma\kappa\nu(\chi_1^0 + \chi_2^0) - 8i\gamma\kappa\lambda\chi_1^0\chi_2^0 &\equiv B_\kappa. \end{aligned}$$

Here $\chi_{1,2}^0$ represent the initial state of the fluctuators. Also of interest are the values of these averaged over the initial states of the fluctuators. Assuming all 4 settings are equally probable we have

$$A_\kappa^{av} = 0, \quad B_\kappa^{av} = \frac{1}{2}\kappa^2(\nu^2 + \frac{1}{2}\lambda^2).$$

In terms of these the coefficients are expressed as

$$\begin{aligned} c_\kappa &= \frac{(\omega_0 - A)(\omega_0 + \omega_+) - (\omega_0^2 - B)}{(\omega_0 - \omega_-)(\omega_+ - \omega_-)}, \\ b_\kappa &= \frac{-(\omega_0 - A)(\omega_0 + \omega_-) + (\omega_0^2 - B)}{(\omega_0 - \omega_+)(\omega_+ - \omega_-)}, \\ a_\kappa &= 1 - b_\kappa - c_\kappa. \end{aligned} \quad (\text{A24})$$

3. General case

The above method is in principle simple to generalize to any number of fluctuators, but the number of equa-

tions increases exponentially in the number of fluctuators.

The general equation is

$$\mathbf{a}_t = M\mathbf{a}, \quad \mathbf{a} = \begin{pmatrix} a \\ b \\ c \\ \vdots \end{pmatrix}$$

Guessing the solution $\mathbf{a} = \mathbf{A}e^{i(\kappa\phi - \omega t)}$ we get the dispersion equation $-i\omega\mathbf{a} = \tilde{M}\mathbf{A}$, which determines the n eigenvalues $\omega_i(\kappa)$ ($i = 1 \dots n$) as functions of κ . Here $n = 2^N$ with N the number of fluctuators. The general solution is

$$\mathbf{a} = \int \frac{dk}{2\pi} e^{i\kappa\phi} A_\kappa e^{\omega t}, \quad e^{\omega t} = \begin{pmatrix} e^{-i\omega_1 t} \\ e^{-i\omega_2 t} \\ \vdots \end{pmatrix}$$

Defining

$$\tilde{\mathbf{a}}_\kappa = \begin{pmatrix} \tilde{a}_\kappa \\ \tilde{b}_\kappa \\ \vdots \end{pmatrix} = A_\kappa e^\omega, \quad \text{and } \tilde{M} = M(\partial_\phi \rightarrow i\kappa)$$

we determine the coefficient matrix A_κ from

$$\underbrace{\left(\tilde{\mathbf{a}}_\kappa(t=0) \mid \tilde{M}\tilde{\mathbf{a}}_\kappa(t=0) \mid \dots \mid \tilde{M}^{n-1}\tilde{\mathbf{a}}_\kappa(t=0) \right)}_{\tilde{M}_a} = A_\kappa \underbrace{\begin{pmatrix} 1 & -i\omega_1 & \dots & (-i\omega_1)^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & -i\omega_n & \dots & (-i\omega_n)^{n-1} \end{pmatrix}}_{\Omega}$$

The matrix \tilde{M}_κ can be written as the contraction of a third order tensor \tilde{M}_T with $\tilde{\mathbf{a}}_\kappa$. Writing the tensor indices we have

$$[\tilde{M}_a]_{ij} = [\tilde{M}_T]_{ijk}[\tilde{\mathbf{a}}_\kappa(0)]_k, \quad [\tilde{M}_T]_{ijk} = [\tilde{M}^{j-1}]_{ij}$$

We then get

$$[A_\kappa]_{ij} = [\tilde{M}_T]_{imk}[\Omega^{-1}]_{mj}[\tilde{\mathbf{a}}_\kappa(0)]_k \equiv [M_\Omega]_{ijk}[\tilde{\mathbf{a}}_\kappa(0)]_k$$

and

$$\begin{aligned} [\tilde{\mathbf{a}}_\kappa(t)]_i &= [A_\kappa]_{ij}[e^{\omega t}]_j = [T_\kappa]_{ik}[\tilde{\mathbf{a}}_\kappa(0)]_k, \\ [T_\kappa]_{ik} &= [M_\Omega]_{ijk}[e^{\omega t}]_j. \end{aligned}$$

One can also write equations for echo experiments. Consider the situation where we initially prepare a state, then apply the echo pulse at time t_e and then measure the state at the time $2t_e$ when the echo signal appears (two

pulse echo). The state just before the application of the echo pulse has to be calculated as above and then this is used as the initial state for the evolution after the echo (it is assumed that the duration of the echo pulse is short). After the echo pulse the matrix \tilde{M} is changed because all jumps of the random walk changes sign. Then also the ω_i change. Let \tilde{M}^- , ω_i^- represent quantities before the echo pulse and \tilde{M}^+ , ω_i^+ after the pulse. Then

$$\begin{aligned} [\tilde{\mathbf{a}}_\kappa(t_e)]_i &= [T_\kappa^-]_{ik}[\tilde{\mathbf{a}}_\kappa(0)]_k \\ [\tilde{\mathbf{a}}_\kappa(2t_e)]_i &= [T_\kappa^+]_{ik}[\tilde{\mathbf{a}}_\kappa(t_e)]_k \end{aligned}$$

with

$$\begin{aligned} [T_\kappa^-]_{ik} &= [M_\Omega^-]_{ijk}[e^{\omega^- t_e}]_j, & [T_\kappa^+]_{ik} &= [M_\Omega^+]_{ijk}[e^{\omega^+ t_e}]_j \\ . \end{aligned}$$

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